Logarithmic OWA Operators in Weighted Averages: Theoretical Advances and Decision-Making Applications

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***Abstract*—Aggregation operators are essential in multi-attribute decision-making, particularly for managing uncertainty and risk. Traditional methods, such as the Ordered Weighted Averaging (OWA) operator, typically address either probability-based weighting or uncertainty-based reordering, but rarely combine both within a unified framework. This paper proposes the Ordered Weighted Logarithmic Averaging Weighted Average (OWLAWA) operator, a novel approach that merges the benefits of weighted averaging and ordered reordering with a logarithmic transformation to better reflect decision-maker preferences under uncertainty. The theoretical properties of this operator including monotonicity, boundedness, and commutativity are formally established. A multi-attribute decision-making framework is then presented, integrating recognized expert weighting methods, including an entropy-based approach, to enhance decision robustness. Through comparative analysis and a sustainability-focused case study involving 20 companies, results demonstrate that the proposed approach yields a controlled sub valuation effect, particularly beneficial in risk-sensitive or compliance-driven environments. These findings indicate a more adaptive and structured decision-making process relative to conventional operators, accommodating both structured probabilities and uncertain preferences. By unifying risk-based and uncertainty-based weighting within a logarithmic formulation, this operator offers a versatile and structured tool for applications in financial risk management, policy evaluation, and supply chain optimization. Future research may explore its integration with fuzzy systems and machine learning methods, further expanding its adaptability in complex decision scenarios.**

***Keywords*—logarithmic aggregation operators, OWA operator, weighted average, generalized mean, sustainability index**

I. INTRODUCTION

Information fusion techniques [1], including aggregation operators [2], have proven highly effective for modeling human-centric decision-making problems. These methodologies find extensive applications across diverse fields from econometrics and finance [3], environmental management [4], production and supply management [5] to sensors and pattern recognition [6]. Their ability to consolidate heterogeneous information sources into a unified decision value makes them particularly suitable for Multi-Criteria Decision-Making (MCDM) contexts [7].

Yager [8] introduced the Ordered Weighted Averaging (OWA) operator. The characteristic reordering mechanism, which enables a parameterized family of operators between the minimum and maximum to be obtained, models decision-maker attitudes, aiding decision-making especially in uncertain scenarios [9]. The OWA operator has led to a plethora of developments in decision-making, such as competitive economic behavior models [10], investment and financial group decision-making [11], multiperson decision-making in health care [12], ranking of tourist destinations based on competitiveness indicators [13] and sustainable development modeling [14], among many others [15].

Recent studies confirm the ongoing relevance of OWA-based methods, including fuzzy measures with Choquet integrals [16], monotone fuzzy inference systems [17], fuzzy MADM with extended OWA weighting [18], Einstein-based fuzzy aggregators [19], and neutrosophic TOPSIS-OWA approaches [20]. Furthermore, OWA-based applications have recently been extended to sustainable decision support systems [21], fuzzy product recommendations [22], and dynamic group decision modeling with advanced fuzzy systems [23].

An interesting development in OWA models is the inclusion of the weighted average in the formulation of the OWA. The mechanism of the Ordered Weighted Averaging Weighted Average (OWAWA) operator [24, 25] integrates information that is bound to the information source using the Weighted Average (WA) and information that is bound to the reordering of the values using the OWA operator. This integration enables the treatment of information under risk and uncertainty in one formula, which is balanced by an integrated importance coefficient. This operator has been extended with several mathematical tools, such as Bonferroni means [26], Heronian means [27], D numbers and linguistic inputs [28], and hybrid intuitionistic fuzzy techniques [29], among others.

Zhou and Chen [30] introduced logarithmic averaging operators. The Generalized Ordered Weighted Logarithmic Averaging (GOWLA) operators extend the decision-making toolset of information fusion techniques. The advantages of these operators, which are based on an optimal deviation technique, originate from their robust mathematical foundation, and a wide-ranging family of these operators have been introduced, e.g., the Pythagorean Fuzzy Induced Ordered Weighted Logarithmic Averaging Distance (PFIOWLAD) operator [31], Induced Ordered Weighted Logarithmic Averaging (IGOWLA) operators [32], Generalized Ordered Weighted Logarithmic Harmonic Averaging (GOWLHA) operators [33], Bonferroni weighted logarithmic averaging distance operator [34] and Generalized Linguistic Weighted Logarithm Averaging (GLWLA) operators [35].

Motivated by these developments and by the identified gap in unifying weighted average mechanisms (risk-based) with OWA (uncertainty-based) through logarithmic transformations, this paper introduces the Ordered Weighted Logarithmic Averaging Weighted Average (OWLAWA) operator. We study its main properties and characteristics. Moreover, we propose a simulation-based technique for further characterizing and identifying the possible advantages of using the OWLAWA operator in a multicriteria decision-making approach. Theoretical contributions are supported with simulation results that highlight OWLAWA's behavior under diverse conditions, and we provide a comparative analysis with existing operators. Finally, an illustrative example is presented for the valuation of sustainable companies, a context strongly aligned with recent decision support applications that involve advanced aggregator models under uncertainty [21, 36].

The remainder of this article is structured as follows. Section II describes the foundations of the study. Section III introduces the proposed OWLAWA operator, its main characteristics, and its properties. Section IV describes a simulation-based technique for the characterization of the OWLAWA mechanism. Section V presents an illustrative example. Finally, Section VI presents the conclusions of this study.

II. Preliminaries

In this section, we examine three operators that motivate the construction of the weighted OWLA operator, namely, the OWA operator, the logarithmic ordered weighted operator and the ordered weighted OWA operator.

A. The Ordered Weighted Averaging Operator

The OWA operator [8] provides a parameterized family of operators that range from the minimum to the maximum of the arguments. The OWA operator is designed to include criterion functions for constructing a global decision function [37] and can be defined as follows:

Definition 1. An OWA operator of dimension n is a mapping OWA: $R^n \rightarrow R$ that includes a weighting vector w such that the sum of the weights is equal to 1 and $w_j \in [0,1]$. The descending formulation of this averaging function is as follows:

$$OWA(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{1}$$

where b_j is the jth largest of the a_i. The arguments can also be ordered in an ascending direction, which depends on the attitude and the decision criteria that are chosen for the assessed problem [38]. The descending OWA operator yields the arithmetic mean when $w_i = \frac{1}{n}$ for all i; when $w = (1,0, \dots 0)$, the descending OWA yields the maximum; and when $w = (0,0, \dots 1)$, the OWA operator returns the minimum [39]. The OWA operator is idempotent and monotonic, and it is bounded and commutative [8, 40].

B. The Generalized Ordered Weighted Logarithmic Averaging Operator (OWLA-GOWLA)

A GOWLA operator is an extension of the Ordered Weighted Geometric Averaging (OWGA) operator that is based an optimal deviation model, which was introduced by Zhou *et al.* [30]. This operator is designed to assess group decision-making problems and is defined as follows:

Definition 2. A GOWLA operator of length vector n is a mapping GOWLA: $\Omega^n \rightarrow \Omega$ that has a characteristic weighting vector w such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$. The value of parameter λ ranges within $(-\infty, \infty)$, according to the following equation:

$$GOWLA(a_1, a_2, \dots, a_n) = \exp\left\{\left(\sum_{j=1}^{n} w_j \left(\ln b_j\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right\}, \tag{2}$$

where, as in the descending OWA operator, the argument b_j is the jth largest of the a_i, which are in decreasing order.

A special case of the GOWLA operator is when the parameter $\lambda = 1$; in this case, we formulate the ordered weighted logarithmic averaging (OWLA) operator [41], which is defined as follows:

Definition 3. An OWLA operator of dimension n constitutes a mapping GOWLA: $\Omega^n \rightarrow \Omega$ with an associated weighting vector of length n such that the sum of the weights is equal to 1 and each weight is between 0 and 1, as follows:

$$OWLA(a_1, a_2, \dots, a_n) = \exp\left\{\sum_{j=1}^{n} w_j \left(\ln b_j\right)\right\}. \tag{3}$$

Following the convention, the arguments b_j are simply ordered from the largest to the smallest of the a_i.

The GOWLA and OWLA operators have been proven to be monotonic, commutative, idempotent, and bounded, please see [30, 41]. Moreover, the ascending and descending GOWLA operators have been distinguished, and by considering diverse formulations of the weighting vector, the maximum, minimum, step, window, Olympic and S-GOWLA operators, among others, have been obtained [30].

C. The OWA Weighted Average Operator

The Ordered Weighted Averaging Weighted Average (OWAWA) operator is a model that unifies the traditional weighted average and the OWA operator. The OWAWA and its induced modeling are introduced in [24, 25]. These formulations enable the assessment of decision-making-based problems under uncertainty and of information under risk; the former are considered using the OWA approach, and the latter are considered using the weighted average as probabilistic input, as follows:

Definition 4. An OWAWA operator of n dimensions is a mapping OWAWA: $R^n \rightarrow R$ that operates with a weighting W vector of dimension n such that the sum of the included weights equals 1 and $w_j \in [0,1]$:

$$OWAWA(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j. \tag{4}$$

Following the convention of the descending OWA operator, b_j is the jth largest of the a_i, and every a_i is accompanied by a WA weight v_i such that $\sum_{j=1}^{n} v_i = 1$ and $v_i \in [0,1]$. $\hat{v}_j = \beta w_j + (1-\beta) v_j$, where $\beta \in [0,1]$

represents the importance of the WA and v_j is the WA weight v_i that follows the reordering b_j, which, in the convention that we follow, is the jth largest of the a_i in decreasing order.

Merigó [24, 25] further introduced a parallel formulation for the OWAWA operator that yields the same result as Eq. (4). However, the elements that affect the WA and the OWA are separated. According to the authors, this formulation does not unify the weighted average and the ordered weighted average models. This representation is defined as follows:

Definition 5. The OWAWA operator of n dimensions is a mapping OWAWA: $R^n \rightarrow R$ with a weighing vector W with components that are between 0 and 1 such that the sum of its components is strictly 1. Additionally, a weighting vector V that follows the same conditions, namely, $\sum_{j=1}^{n} v_i = 1$ and $v_i \in [0,1]$, is included, this mapping is expressed as follows:

$$OWAWA(a_1, a_2, \dots, a_n) = \beta \sum_{j=1}^{n} w_j b_j + (1-\beta) \sum_{i=1}^{n} v_i a_i, \tag{5}$$

where $\beta \in [0,1]$ represents the degree of importance of the WA and b_j is the jth largest of the a_i.

The OWAWA operator follows the OWA operator in being monotonic, commutative, bounded and idempotent [25], and it can be formulated as a descending OWAWA operator or an ascending OWAWA operator, depending on the reordering process of the arguments. These modifications would only affect the weighting vector W.

III. The OWLAWA Operator

Motivated by the advancements of Zhou and Chen [30] and Merigó [24, 25], this section presents the Ordered Weighted Logarithmic Averaging Weighted Average (OWLAWA) operator. The main advantage of this operator is the unification of the weighted average and the OWA operator using logarithmic averaging functions. Thus, we extend the current tools for decision-making problem assessment under uncertainty and probabilistic conditions. The OWLAWA operator is defined as follows:

Definition 6. An OWLAWA operator of dimension n is a mapping OWLAWA: $\Omega^n \times \Omega^n \rightarrow \Omega$ that has an associated weighting vector W such that the sum of its components is equal to 1 and $w_j \in [0,1]$, which is expressed as follows:

$$OWLAWA(a_1, a_2, \dots, a_n) = \exp \sum_{j=1}^{n} \hat{v}(\ln b_j), \tag{6}$$

where b_j is the jth largest of the a_i and each argument a_i has an associated weight (WA) v_i such that the sum of its elements is equal to 1 and $v_i \in [0, 1]$. Here, $\hat{v} = \beta w_j + (1-\beta) v_j$, $\beta \in [0, 1]$ and v_j is the weight (WA); v_i is ordered according to b_j, namely, according to the jth largest of the a_i.

The OWLAWA operator can also be separated into two sections: the part that strictly affects the OWLA operator and the WA. This formulation results in a more straightforward approach and is defined as follows:

Definition 7. An OWLAWA operator is a mapping OWLAWA: $\Omega^n \times \Omega^n \rightarrow \Omega$ that includes an associated weighting vector W such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$ and a vector v_i such that $\sum_{j=1}^{n} v_i = 1$ and $v_i \in [0,1]$, which is expressed as follows:

$$OWLAWA(a_1 \dots a_n) = \exp\left\{\beta \times \left(\sum_{j=1}^{n} w_j (\ln b_j)\right) + (1-\beta) \times \left(\sum_{i=1}^{n} v_i (\ln a_i)\right)\right\}, \tag{7}$$

where b_j is the jth largest a_i in descending order. Both Eqs. (6) and (7) yield the same result.

Let us briefly demonstrate the aggregation process using the OWLAWA operator. In this example, we utilize both definitions.

Example 1. Consider arguments $a_i = (51, 26, 37, 42)$ to be introduced in the aggregation process, a weighting vector W of $(0.2, 0.1, 0.3, 0.4)$ and a WA weighting vector V of $(0.5, 0.3, 0.1, 0.1)$. Suppose the importance degree of the WA is 60%, namely, that of the OWLA aggregation is 40%. Here, calculate the associated weighting vectors for Eq. (6):

$$\hat{v}_1 = 0.4 \times 0.2 + 0.6 \times 0.5 = 0.38,$$
$$\hat{v}_2 = 0.4 \times 0.1 + 0.6 \times 0.1 = 0.10,$$
$$\hat{v}_3 = 0.4 \times 0.3 + 0.6 \times 0.1 = 0.18,$$
$$\hat{v}_4 = 0.4 \times 0.4 + 0.6 \times 0.3 = 0.34.$$

Next, by Eq. (6), we obtain:

$$OWLAWA = \exp\{0.38 \times (\ln 51) + 0.1 \times (\ln 42) + 0.18 \times (\ln 37) + 0.34 \times (\ln 26)\} = 37.55$$

In this case, we opt to calculate the OWLAWA operator with Eq. (7):

$$OWLAWA = \exp\{0.4 \times \{0.2 \times (\ln 51) + 0.1 \times (\ln 42) + 0.3 \times (\ln 37) + 0.4 \times (\ln 26)\} + 0.6 \times \{0.5 \times \ln(51) + 0.3 \times \ln(26) + 0.1 \times (\ln 37) + 0.1 \times \ln(42)\}\} = 37.55.$$

The OWLAWA operator shares the properties of monotonicity, boundedness and commutativity with the OWAWA [24] and OWLA [30, 41] operators. Let us briefly explore these properties with the following theorems.

Theorem 1 (Monotonicity). Suppose f is an OWLAWA operator. If $a_i \geq g_i$ for all $i \in \{1, 2, \dots, n\}$, then:

$$f(a_1, a_2, \dots, a_n) \geq f(g_1, g_2, \dots, g_n). \tag{8}$$

The proof is straightforward and, thus, is omitted.

Theorem 2 (Boundedness). Let f be an OWLAWA operator. If $\max_i a_i = a_{max}$ and $\min_i a_i = a_{min}$, then:

$$a_{min} \leq f(a_1, a_2, \dots, a_n) \leq a_{max}, \tag{9}$$

considering

$$f(a_1, a_2, \ldots, a_n) = \exp \sum_{j=1}^{n} \hat{v}_j(\ln b_j), \tag{10}$$

When $\max_i a_i = a_{max}$, then $b_j \leq a_{max}$ for every j. Therefore,

$$f(a_1, a_2, \ldots, a_n) = \exp \sum_{j=1}^{n} \hat{v}_j(\ln b_j) = a_{max}. \tag{11}$$

Similarly, $f(a_1, a_2, \ldots, a_3) \geq a_{min}$.

$$a_{min} \leq f(a_1, a_2, \ldots, a_3) \leq a_{max}, \tag{12}$$

This proof is completed by assigning full importance to the OWLA. As this might not always be the case, we explore the following:

Theorem 4 (Semi boundary conditions). Here, we define f as an OWLAWA operator. Then,

$$\begin{aligned}\exp\left\{\beta \times a_{min} + (1-\beta) \times \sum_{i=1}^{n} w_i(\ln a_i)\right\} \\ \leq f(a_1, a_2, \ldots, a_n) \\ \leq \exp\left\{\beta \times a_{max} \right. \\ \left. + (1-\beta) \times \sum_{i=1}^{n} w_i(\ln a_i)\right\},\end{aligned} \tag{13}$$

where, in the case of $\beta = 1$, the traditional boundary conditions are satisfied. In addition, from the OWLAWA perspective, we have:

$$\begin{aligned}\exp\left\{\beta \times a_{min} + (1-\beta) \times \sum_{j=1}^{n} w_j(\ln b_j)\right\} \\ \leq f(a_1, a_2, \ldots, a_n) \\ \leq \exp\left\{\beta \times a_{max} \right. \\ \left. + (1-\beta) \times \sum_{j=1}^{n} w_j(\ln b_j)\right\},\end{aligned} \tag{14}$$

The proof follows similarly to the previous case.

Theorem 5 (Commutativity). Suppose f is an OWLAWA operator, where $(c_1, c_2, \ldots, c_n)$ corresponds to any permutation of the arguments a_i, namely, $(a_1, a_2, \ldots, a_n)$. Since $(c_1, c_2, \ldots, c_n)$ is a permutation of $(a_1, a_2, \ldots, a_n)$, $b_j \geq d_j$ for all j. Hence,

$$f(a_1, a_2, \ldots, a_n) \geq f(c_1, c_2, \ldots, c_n). \tag{15}$$

The proof is straightforward.

The OWLAWA operator can be regarded as a unification of the weighted average and the logarithmic aggregation operators. This model can also be constructed under similar approaches, e.g., the weighted OWA operator [42], which enables the aggregation of a set under two weighting vectors and, hence, the weighting of the reliability of the data inputs, namely, weighted average inputs, and their values according to their relative positions or the OWA of other values. Another formulation can be obtained if we consider the Hybrid Weighted Averaging (HWA) operator [43], which was designed to combine the advantages of the weighted arithmetic averaging and OWA operators using a balancing coefficient. With the premise of immediate probabilities for decision-making on a set of alternatives, we can also construct a different formulation of the OWLAWA operator, in this case following [44–46]. Here, we could consider that the perception of the possible scenarios can be influenced by the associated payoffs and the outcome of a decision. Since we can extend the immediate probabilities to a weighted average, we could use the immediate weighted average for unification of the OWLA operators. Finally, we also consider a method that is based on importance weights [47], which is an approach that involves the transformation of scores into an effective value according to their respective importance, similar to the weighted average inclusion of immediate probabilities.

Interesting cases are constructed when analyzing the components of the OWLAWA operator. These formulations also provide a general landscape of the applications of these operators.

Remark 1. For the importance coefficient β, we observe the following:

If $\beta = 0$, we obtain the weighted average.

If $\beta = 1$, we construct the OWLA operator.

If β is increased, more importance is given to the OWLA operator; correspondingly, if it is decreased, more importance is given to the WA.

Remark 2. For the W-associated weighting vector, we also obtain interesting results:

In the case that $\beta = 1$, $w_1 = 1$ and $w_j = 0$, for every j, we obtain $\max\{a_i\}$.

If $\beta = 1$, $w_n = 1$ and $w_j = 0$, for every $j \neq n$, we obtain $\min\{a_i\}$.

For $\beta = 0$, $w_1 = 1$ and $w_j = 0$ and for all j, $\min\{a_i\}$ is obtained.

For $\beta = 0$, $w_n = 1$ and $w_j = 0$, for all $j \neq n$, $\max\{a_i\}$ is obtained.

The Olympic OWLAWA operator is obtained when $w_1 = w_n = 0$ and $w_j = \frac{1}{(n-2)}$.

For a general perspective, when $w_k = 1$ and $w_j = 0$ for every $j \neq k$, we obtain the step-OWLAWA operator.

Remark 3. A weighted logarithmic averaging operator can also be obtained when $w_j = \frac{1}{n}$ for all j. Here, we obtain this formulation as follows:

$$\begin{aligned}WLA(a_1, a_2, \ldots, a_n) \\ = \frac{1}{n}\beta \sum_{i=1}^{n} a_i \\ + (1-\beta) \sum_{i=1}^{n} v_i a_i.\end{aligned} \tag{16}$$

In the case of $v_i = \frac{1}{n}$ for every i, an ordered logarithm average is obtained:

$$\begin{aligned}OWLA(a_1, a_2, \ldots, a_n) \\ = \beta \sum_{i=1}^{n} w_j b_j + \frac{(1-\beta)}{n} \sum_{i=1}^{n} a_i.\end{aligned} \tag{17}$$

Several other formulations can also be constructed following [24, 30, 40, 48] for the associated weighting vectors W and V, along with the representation of the combinations when applying the importance coefficient β.

IV. Characterization of the Behavior of the OWLAWA Operator

The behavior of the OWLAWA operator can be further characterized using simulation techniques. In this exercise, we compare the OWA, OWAWA, OWLA and OWLAWA operators. The same initial conditions are used for each operator, and the results are plotted to further characterize the behavior of the proposed OWLAWA operator.

A total of 1000 iterations are run to represent the characteristic behaviors of the operators. The initial conditions of the test are presented in Table 1.

Table 1. Initial conditions for the characterization of the behavior of the OWLAWA operator

Element	Initial condition
Iterations	$n = 1000$
Arguments	$a_i = (11, 52, 46, 27, 88)$
Importance coefficient	$\beta = 0.3$
Weighting vector W	$\frac{\sum_{j=1}^{n} w_j}{n} = (0.124, 0.126, 0.249, 0.252, 0.249)$
Weighting vector V	$\frac{\sum_{i=1}^{n} v_i}{n} = (0.353, 0.347, 0.100, 0.099, 0.101)$

As the iterations proceed for the simulation of the OWA weights, Table 1 presents the average value of each weighting vector.

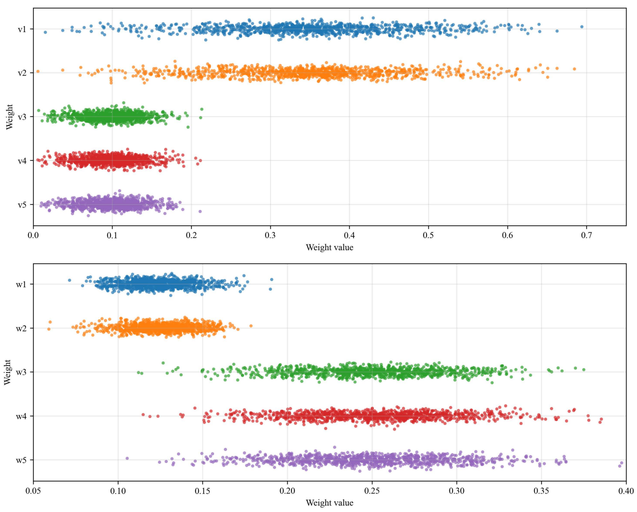

Fig. 1. Description of the weights v_i and w_j.

The complete dataset for weights w_j and v_i is presented in the appendix of this paper. We observe that the weighting vector W is pessimistic, as weights w_{3-5} are heavier than $w_{1,2}$, while in weighting vector V, the first two weights are heavier. Fig. 1 shows the distributions of the iterated weights.

Using Eqs. (1), (3), (4), and (6), the traditional weighted average for the arguments and the initial conditions that are presented in Table 1, we compute the WA, OWA, OWLA, OWAWA and OWLAWA operators for each iteration. The results are shown in Fig. 2. The aim is to characterize the behavior of each operator and, with the resulting differences, observe possible phenomena and scenarios where the application of the OWLAWA operator can be useful. The importance coefficient is $\beta = 0.3$; thus, we give higher importance to the WA than to the OWLA aggregation.

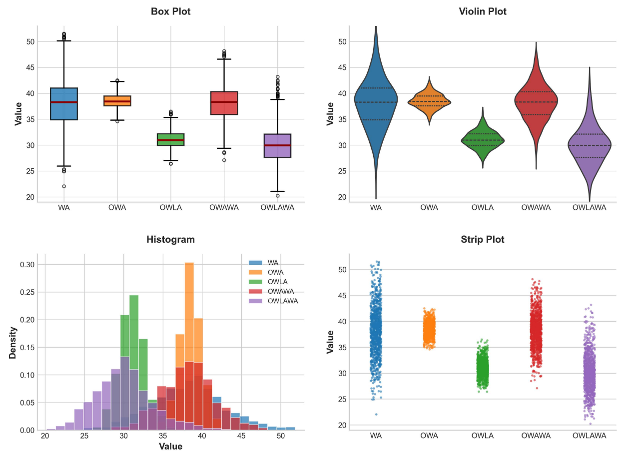

Fig. 2. WA, OWA, OWLA, OWAWA and OWLAWA results for $n = 1000$ iterations.

The results show that for the total number of iterations, the operators behave similarly; however, due to the characteristic mathematical composition of each operator, the results vary. The OWA and the OWLA operators behave similarly, as do the OWAWA and the OWLAWA operators, namely, the logarithmic operators undervalue the general score compared to the nonlogarithmic operators. Additionally, when using weighted averaging OWA operators, the distribution of the values is wider. This is easily observed in Fig. 1, in which histogram of the scores is presented. This shows that the values of each of the nonweighted operators are within a narrower range whereas those of each of the weighted operators are distributed over a wider range.

This exercise is not exhaustive, as the initial conditions can have a plethora of compositions. Nonetheless, the characteristic behavior of the logarithmic operators is interesting to analyze, as the undervaluing of the scores and, for the weighted averaging OWLA operators, the wider distribution of the results can be used for decision-making applications in which these characteristics are required.

V. Illustrative Example

The characteristic properties of the proposed OWLAWA operator make it interesting for a wide-ranging set of applications, e.g., in statistics, engineering, soft computing, business, management and financial decision-making; for a comprehensive discussion of areas in which aggregation operators have proven to be effective [2].

In this paper, we focus on a multicriteria decision-making application for the appraisal of sustainable companies. The objective is to quantify the efforts that some businesses employ for sustainable actions. This quantification enables the ranking and indexing of companies that are considered sustainable. The OWLAWA operator is suitable for this application, as the characteristic undervaluing of the aggregation score can be used for a rigorous approach. Additionally, the integrated WA mechanism enables an assessment of the given data, which were obtained probabilistically. For this illustrative example, we focus on the assessment of experts in the field of sustainability, whose valuations of each business are aggregated to yield a grade or rating of the assessed company. This approach is especially relevant when the available information is characterized by uncertainty.

The application of the OWLAWA operator requires an initial set of arguments and valuations for the aggregation to enable effective decision-making. The application of a multicriteria decision-making approach using the proposed operator is described by the following steps:

Step 1. First, a set of $C = \{c_1, c_2, \dots, c_m\}$ finite alternatives must be defined; in our case, these are companies. For these, a set $S = \{s_1, s_2, \dots, s_n\}$ of limited states or attributes of sustainability are used to construct a payoff matrix $(a_{hi})_{m\times n}$. In this case, we define a limited set of experts $E = \{e_1, e_2, \dots, e_p\}$ and a weighting vector $X = \{x_1, x_2, \dots, x_p\}$ such that the sum of the weights is equal to 1 and $x_k \in [0,1]$. The weighting vector X corresponds to the experts' influence in the aggregation process. The experts' opinions should be in the range of $y_r \in [1, 100]$ for $r = 1,2, \dots p$. For each expert, a valuation matrix $\left(a_{hi}^{(k)}\right)_{m\times n}$ is required.

A. Expert Weight Determination via the Entropy Method

To enhance the robustness and objectivity of the proposed decision-making model, we introduce an entropy-based approach for determining the weights assigned to expert opinions [49]. This method complements the initial subjective weight assignment (based on expertise and field relevance) and is well-suited to contexts involving uncertainty and heterogeneous judgments conditions under which OWLAWA is particularly effective.

The Entropy Method [50] has been widely adopted in multi-attribute decision-making (MADM) models to evaluate the degree of information provided by each source [51]. Experts whose assessments vary little across alternatives are considered to convey less useful information and are thus assigned lower weights. Conversely, experts whose evaluations show higher variability contribute more to the decision-making process and are given higher importance.

Let x_{ij} be the score assigned by expert e_j to alternative a_i, and the normalized value is:

$$r_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}}. \tag{18}$$

Then, the entropy E_j for each expert is calculated as:

$$E_j = -k \sum_{i=1}^{m} r_{ij} \cdot ln(r_{ij}), \tag{19}$$

where

$$k = \frac{1}{ln(m)}. \tag{20}$$

The divergence degree is:

$$d_j = 1 - E_j, \tag{21}$$

Finally, the normalized weight w_j of expert e_j is:

$$w_j = \frac{d_j}{\sum_{j=1}^{n} d_j}. \tag{22}$$

This approach yields a reproducible and mathematically grounded vector of expert weights that can be integrated into the OWLAWA operator. In this study, we retain the initially defined weights (0.5, 0.3, 0.2) based on domain knowledge, but the entropy-based weighting is presented here as a generalizable alternative for future applications of the model.

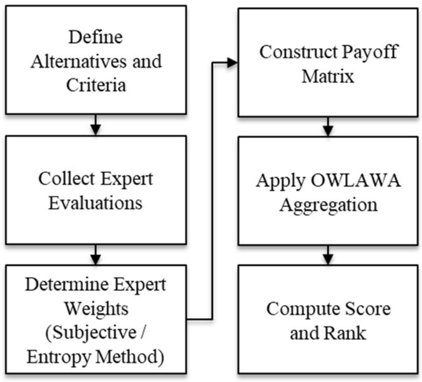

Fig. 3. Flowchart of the OWLAWA-based multi-attribute decision-making model.

Fig. 3 presents a flowchart representation of the proposed multi-attribute decision-making model using the OWLAWA operator.

The process begins by defining a set of alternatives and sustainability attributes, followed by the collection of evaluations from multiple experts. The expert weights are then assigned either subjectively (based on experience and relevance) or objectively through the Entropy Method. These inputs form a collective payoff matrix, which is then aggregated using the OWLAWA operator to compute a sustainability score for each alternative. The final step involves comparing the results to a predefined threshold or performing a ranking to identify sustainable companies.

Step 2. Define the weighting vector $\hat{V} = W + (1 - \beta)V$, which expresses the characteristic approach of the OWLAWA operator. The importance coefficient β corresponds to the uncertainty of the retrieved data, and the selection depends on the analyzed phenomena. Additionally, $W = (w_1, w_2, \dots, w_3)$ satisfies $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, and $V = (v_1, v_2, \dots, v_3)$ satisfies $\sum_{j=1}^{n} v_i = 1$ and $v_i \in [0,1]$.

Step 3. Aggregate the valuations of the set of experts E with the weighting vector X and establish a collective payoff $(a_{hi})_{m\times n}$, where $a_{hi} = \sum_{k=1}^{p} x_k a_{hi}^{k}$.

Step 4. Calculate Eq. (6) to obtain the OWLAWA operator. Many formulations can be constructed depending on the families and particular cases of the operator.

Step 5. Establish a parameter Q that defines the minimum score for being considered a sustainable company. In this case, $Q \in [0,100]$ is defined by decision-makers, and it should be designed to assess the conditions of the environment, e.g., the industry, number of workers, and regional conditions. Every company C such that $OWLAWA_n \geq Q$ is indexed in the sustainable company category.

Step 6. Decision-making approach. We compare the results with those of other formulations, operators, and families of the OWLAWA operator to categorize and rank the evaluated companies.

B. Numerical Example

Let us illustrate the proposed decision-making approach with a numerical example. The exercise focuses on the application of the OWLAWA operator to a series of companies for which sustainable actions must be quantified for the creation of a sustainability index of companies. This index enables the categorization, hierarchical ordering and ranking of the evaluated companies; moreover, the results of the evaluation can be used to develop public and economic policies and incentives. The following steps are implemented:

Step 1. For the large industrial manufacturing sector of a city, let us evaluate a set of 20 companies; thus, $C = \{c_1, c_2, \ldots, c_{20}\}$. These companies will be ranked in terms of sustainable practices in the following areas:

S1: Economic performance
S2: Market share
S3: Unfair competition
S4: Supplies
S5: Energy consumption management
S6: Water consumption
S7: Environmental impact management
S8: Gas emissions
S9: Labor relations
S10: Community impact

The evaluation is carried out by three experts $E = \{e_1, e_2, e_3\}$, whose opinions will be expressed in the range of 1 to 100, where 1 is the lowest possible score and 100 is the maximum possible score. The expert opinion weighting vector X is set to (0.5, 0.3, 0.2), which corresponds to their experience and fields of expertise. Table 2 presents the collective aggregated opinions of the three experts.

Step 2. The decision-makers require an OWLA weighting vector W = (0.0954, 0.1155, 0.1407, 0.0087, 0.0634, 0.1085, 0.1218, 0.1188, 0.1127, 0.1143), a WA weighting vector V = (0.0416, 0.1465, 0.1061, 0.1291, 0.0186, 0.0888, 0.1428, 0.0285, 0.1484, 0.1495) and a coefficient of importance $\beta = 30\%$. Thus, the WA is assigned a weight of 70% and the OWLA a weight of 30%.

Step 3. With this information, we calculate the expert payoff matrix. Table 2 shows the results of the valuations; the individual valuations of the experts are presented in the appendix of this paper.

To evaluate the applicability of the proposed operator, we conducted a comparative analysis involving four classical aggregation methods—WA, OWA, OWLA, and OWAWA—alongside OWLAWA. The outcomes are summarized in Table 3 and visually represented in Figs. 4 and 5, highlighting their distinctive behaviors.

Step 4. The collective payoff matrix and the initial information enable the construction of diverse aggregations. For this case, we calculate the WA, the OWA, OWLA, OWAWA and OWLAWA operators. Table 3 presents the aggregated results for each company.

Step 5. Due to the characteristics of the industry, regional practices, and experts' opinions, a threshold of $Q = 68$ is set. Therefore, considering the OWLAWA operator, the companies that are indexed as sustainable are as follows: Sustainable companies = $\{c_1, c_7, c_9, c_{13}, c_{16}, c_{20}\}$.

Table 2. Collective payoff matrix

Company	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10
C1	91.6	91.6	86.9	88.9	85.3	78.3	87.9	85.4	79.1	87.9
C2	56.3	76.8	56	71.7	56.3	58.3	66	67.9	61.6	67.1
C3	60.1	49.5	72	56.8	62.7	70.3	45.8	76	73.7	70.5
C4	69.4	72	66.2	55.9	70.8	55.4	68.9	65.8	61.7	60.8
C5	39.1	51.9	23	44.4	41.4	31.4	37.5	58.9	25.7	35.9
C6	61.2	71.2	70.1	79.5	63	54.1	74	59.6	67.3	49.7
C7	86.2	89.3	90.8	95.2	89.3	77.7	85.7	83.8	77.3	91.2
C8	76.3	74.3	76.2	55.4	64.5	65.8	62.9	66.9	61.2	56.3
C9	78	78.8	90	91	81.5	90.3	76.6	88.3	85.3	89.7
C10	72.2	77.1	66.1	70.3	56.6	52.9	62.8	55	57	68.5
C11	76.8	54.6	67.1	69.4	70.9	43.5	62.2	50.3	57.1	54.5
C12	70.7	54.3	67.7	69.2	60.3	67.8	55	68.1	66.4	65.3
C13	65.5	67.2	71.9	65.6	66.1	61.3	62.9	65.9	69.1	64.8
C14	77.5	68.2	61.6	57.3	79.5	52.8	74.3	60.1	58.1	67.6
C15	32.6	52.9	30.2	44.9	27.5	39.7	55.5	27.5	30.6	41.1
C16	74.3	79.5	93.2	78.6	94.9	72.3	91.8	83	85.6	82.6
C17	43.4	49.1	42.3	25.8	47.3	30.4	27.4	47.5	44.9	44.8
C18	49.7	70.2	62.9	67.8	58	63.7	49.8	69.7	60.1	59.1
C19	67.3	80.5	63.1	65.9	74.2	88.8	53.5	65.2	67.6	77.1
C20	89.4	84.1	77.1	91	86.4	86.9	78.8	76.5	82.9	87.7

Table 3. Comparative aggregation results using WA, OWA, OWLA, OWAWA, and OWLAWA

Company	WA	OWA	OWLA	OWAWA	OWLAWA
C1	81.53	81.16	81.02	81.42	81.27
C2	65.77	65.32	64.97	65.64	65.28
C3	63.46	64.28	63.67	63.71	63.04
C4	65.70	64.48	64.25	65.34	65.13
C5	31.78	32.92	31.73	32.12	30.83
C6	68.23	68.52	68.04	68.32	67.81
C7	83.36	84.41	84.29	83.68	83.59
C8	69.24	66.34	65.32	68.37	67.51
C9	84.67	84.97	84.74	84.76	84.56
C10	63.65	63.43	63.27	63.58	63.41
C11	69.09	66.64	66.02	68.36	67.75
C12	67.32	64.54	64.14	66.49	66.11
C13	70.87	69.15	68.95	70.36	70.19
C14	65.04	62.93	62.65	64.41	64.11
C15	36.04	33.32	30.79	35.22	32.58
C16	85.37	86.06	85.92	85.58	85.43
C17	40.48	38.80	37.17	39.97	38.30
C18	64.48	60.74	59.96	63.35	62.58
C19	62.99	64.47	63.64	63.43	62.59
C20	85.90	83.74	83.44	85.25	84.97

Fig. 4 shows a graphical representation of the valuation of each company. The general undervaluing of the aggregation score by the OWLAWA operator has an impact on the sustainable company index, specifically for $c_6, c_8,$ and c_{11}. Fig. 5 more closely examines $c_6, c_8,$ and c_{11}. Here, we observe that the undervaluing of the aggregation score by the OWLAWA operator results in the exclusion of 3 companies that are included in the index when evaluated by the OWAWA operator. This undervaluing acts as a rigorous approach for the decision-making process, and in some cases, this characteristic can be of interest in the selection and application of this operator.

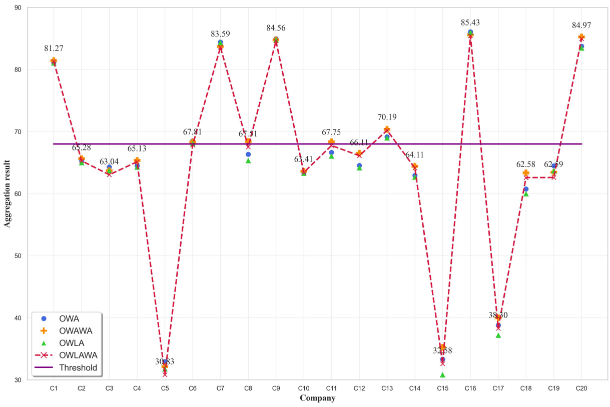

Fig. 4. Graphical representation of the results.

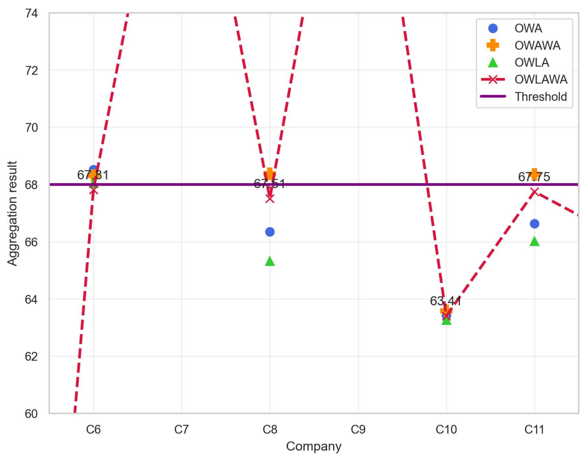

Fig. 5. In depth observation of the evaluation of c_6, c_8, and c_{11}.

These findings suggest that the OWLAWA operator may be particularly suitable for conservative applications where the underestimation of performance acts as a safeguard—such as sustainability compliance, risk assessment, or regulatory filtering. Moreover, the decision-making approach differs among the selected operators, namely, the result of each operator is different. The proposed application is interesting when multicriteria decision-making is required for the assessment of elements that have information under risk and uncertainty and when a rigorous approach is desired and a threshold is set. Other approaches can also be applied, for example, a ranking of the companies based on the selected operators or a similar strategy. The selected decision-making approach follows the conditions of the studied phenomena.

VI. Conclusion

The objective of this paper was the proposal of the ordered weighted logarithmic averaging weighted average (OWLAWA) operator. The characteristic design of this operator enables the treatment of information under uncertainty and risk in one formulation. The OWLAWA operator is based on the optimal deviation model, which was developed by Zhou and Chen [30], and shares its main properties. We explored some of the main characteristics and families of the OWLAWA operator and described various particular cases and compositions when analyzing the weighting vector W.

We also further explored the characteristic design of the OWLAWA operator using simulation techniques. The exercise included a total of 1000 simulations of aggregations with similar initial conditions for the WA, OWA, OWLA and OWLAWA operators. The results show that the logarithmic averaging operators undervalue the aggregation score in general, and the weighted average values are spread over a wide range for the operators that include this feature. These observed characteristics will be interesting to further explore in future research.

We also presented an illustrative example of a multicriteria decision-making problem using the features of the OWLAWA operator. In the example, companies' efforts for sustainability were evaluated. Three experts assigned scores to the companies, and if the aggregation score of a company surpassed an established threshold, the company was indexed as a sustainable company. The observed features of the OWLAWA operator are desirable because the logarithmic operators generally undervalue the aggregation score, which can be used as a rigorous criterion. The results show that when using the OWLAWA operator, 6 companies were indexed as sustainable. In contrast, when the OWAWA operator was used for this assessment, a total of 9 companies were indexed, namely, the use of the OWLAWA operator resulted in the aggregation of fewer indexed companies.

The main objective of the OWLAWA operator is the combination of risk and uncertain information into a single formulation using an importance coefficient and logarithmic averaging operators. Since the proposal of the GOWLA operators, many developments and applications have been reported. Therefore, the proposal of an extended toolset for decision-making processes is of interest, as the real-world challenges continue to increase in complexity.

From a theoretical standpoint, OWLAWA extends traditional aggregation models by providing a framework that simultaneously handles both probabilistic and uncertain information. This dual-layered approach contributes to decision science by allowing for more flexible information fusion, adapting to different decision-maker risk attitudes. Furthermore, logarithmic transformation introduces a controlled sub valuation effect, which is particularly useful for conservative decision environments. Compared to existing OWA-based operators, OWLAWA ensures that aggregation results account for both structured probabilities and uncertain preferences in a mathematically coherent way.

The OWLAWA operator has strong potential for application in diverse decision-making domains. Beyond sustainability assessment, where it was demonstrated in this study, it can be extended to financial risk analysis, supply chain optimization, healthcare prioritization, and policy-making under uncertainty. One of its key advantages is the ability to incorporate expert judgments dynamically through adaptive weighting methods and entropy-based approaches, thereby enhancing decision robustness in complex scenarios. Additionally, OWLAWA provides an adaptable framework for multi-criteria decision analysis, allowing fine-tuned control over aggregation processes depending on the nature of the available data.

This study provides a solid foundation for the application of the OWLAWA operator; however, several avenues remain open for further exploration. The simulation-based characterization employed here offers controlled insights, yet future studies could enrich these findings by incorporating real-world datasets with more diverse uncertainty profiles. The sustainability case study illustrates the operator's utility in practice, but additional applications in domains such as finance, healthcare, or policy analysis would help validate its broader adaptability. The current model assumes consistent expert judgments; extending the framework to address conflicting or fuzzy expert input could enhance its robustness. Lastly, although our comparative analysis involved five well-established operators under the same conditions, exploring OWLAWA's performance across other decision-making scenarios could further demonstrate its versatility and strength.

Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

Gerardo G. Alfaro-Calderón wrote the manuscript, developed the mathematical derivations, and conducted the computational experiments. Víctor G. Alfaro-García drafted the study together with José M. Merigó, carried out the data analysis and simulations, generated the figures and comparative evaluations, and coordinated revisions as corresponding author. José M. Merigó co-conceived the study, provided the theoretical foundation for the OWLAWA operator, and verified the mathematical formulations. All authors contributed to the interpretation of results, critically revised the manuscript, and approved the final version.

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