Discrete-Time Output Regulation on Sample-Data Systems

Muwahida Liaquat and Mohammad Bilal Malik

***Abstract*—In this paper output regulation problem for sampled-data systems with constant exogenous signals is considered. A discrete output feedback controller is designed under the assumption that the regulator equation in discrete-time has a solution. Discretization of the plant and exosystem is carried out by zero order hold equivalence. The goal is to design the complete controller/observer in discrete domain. As an application magnetic levitation is discussed. The simulation results along with the comparison with the continuous-time controller show the effectiveness of the proposed controller.**

***Index Terms*—Discretization, linear systems, output regulation, sampled-data control.**

I. Introduction

Output regulation problem is a fundamental problem in control theory. It has been studied very actively since the seminal work by Francis in 1977 [1], and up till now, we have quite a complete solution. It addresses the problem of designing a feedback controller to achieve asymptotic tracking for a class of reference inputs and rejection for a class of disturbances while maintaining closed loop stability. The theory of LTI systems is well known [1-3]. It has been extended to time-varying systems, where instead of using classical (algebraic) regulator equations differential regulator equations are applied [4-6]. This problem was then extended to its corresponding problem in non-linear setting [7]. For sampled-data systems with zero order holds, the problem of output regulation with constant exogenous signals is easily solved [8-9] but the problem complicates for general exogenous signals as the ripples between the sampling period complicates matter and continues-time pre-compensators are necessary to attenuate ripples [9]. Generalized hold devices are presented in [10-11].

In navigation and control of auto-motives such as vehicles, ships and airplanes, control of attitude is important and tracking of constant signals is considered. Output regulation problem for sampled-data systems is given by (1).

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t) \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} \tilde{u}(t) \end{aligned} \qquad (1)$$

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The authors are with the Department of Electrical Engineering, College of Electrical and Mechanical Engineering, National University of Sciences and Technology, Pakistan (e-mail: muwahida@ee.ceme.edu.pk, mbmalik@ceme.edu.pk).

where $x \in R^n$ is the state, $\tilde{u} \in R^{m2}$ is the control input realized through a zero-order hold, i.e. $\tilde{u}(t) = u[k], k\tau \le t < (k+1)\tau$. Here τ is the sampling time and $z \in R^{p1}$ is the output to be regulated. The signal $y \in R^{p2}$ is the sampled measurement realized through a zero-order hold and is given as:

$$y[k\tau] = C_2 x[k\tau] \qquad (2)$$

The constant exogenous signal $w \in R^{m1}$ is generated by

$$\dot{w}(t) = 0 \qquad (3)$$

For the sampled-data system (1) and (3) we want to find a discrete time controller such that the closed loop system is asymptotically stable and the following condition is fulfilled: $\lim_{t\to\infty} z(t) = 0,$ for any $x(0)$ and $w(0).$

In the typical output regulator the observer and state feedback gains can be designed independently by using the separation principle. Two approaches have conventionally been used to design a sampled-data control system. In the first approach, a continuous-time controller/observer is designed that stabilizes the continuous-time system model. The controller/observer is then discretized and implemented. This approach has been applied in [12] where a discrete-time output feedback controller is designed under the assumption that the regulator equation in the continuous-time has a solution. The second approach is based on finding some discrete-time equivalent model for the continuous-time system to be stabilized. The controller/observer design is subsequently carried out completely in the discrete domain.

Since, the closed form exact discrete-time model for continuous-time system required for controller/ observer design can be obtained conveniently for linear time-invariant systems, in general [13] we have designed a discrete-time output feedback controller based on the latter approach. The continuous-time system and the exosystem are first discretized via zero-order hold and then the solution to the regulator equations in discrete-time is computed to design the corresponding discrete-time controller. In this way the overall system can be presented in discrete-time.

Section 1 is introduction and Section 2 gives assumptions and the main result. Performance of the designed controller is illustrated in Section 3 by applying this control to a single link robot and Section 4 is conclusion.

Notations: Throughout this paper, let $\sigma(M)$ be the set of all Eigen values of a square matrix M .

II. Output Regulation for Sampled Data Systems

Consider the sampled-data system (1) and the exosystem (2) with the following assumptions:

1) (A_d, B_{2d}) is stabilizable.
2) There exist matrices Π and Γ which satisfy the regulator equation

$$\begin{aligned}\Pi S_d &= A_d\Pi + B_{1d} + B_{2d}\Gamma \\ 0 &= C_{1d}\Pi + D_{11d} + D_{12d}\Gamma\end{aligned}$$

3) $(C_{2e}, e^{A_e\tau})$ is detectable.

Let $A_d = e^{A\tau}$, $B_{id} = \int_0^\tau e^{Ar}dr\, B_i$ where $i = 1,2$ and let $f_\tau(A) = \int_0^\tau e^{Ar}dr$. We have $B_{id} = f_\tau(A)B_i$ for i=1,2 and $A_d - I = f_\tau(A)A$. By [17], we have that matrix $f_\tau(A)$ is invertible if (A_d, B_{2d}) is stabilizable. Then by combining the results in [12] and [9] we have the following result.

A. Theorem 2.1:

Consider the system (1) and (2) and if the assumptions A1-A3 hold then the output feedback controller

$$\begin{bmatrix}\hat{x}\\ \hat{w}\end{bmatrix}[k+1] = e^{A_e\tau}\begin{bmatrix}\hat{x}\\ \hat{w}\end{bmatrix}[k] + f_\tau(A_e)\begin{bmatrix}B_2\\ 0\end{bmatrix}u[k] + K\{y[k] - C_{2e}\begin{bmatrix}\hat{x}\\ \hat{w}\end{bmatrix}[k]\} \quad (4)$$

$$u[k] = \begin{bmatrix}F & \Gamma - F\Pi\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{w}\end{bmatrix}[k]$$

fulfills output regulation for the sampled-data system (1) where F is chosen such that $\sigma(A, B_2) \subset C^-$ and $\sigma(A_d, B_{2d}) \subset D$. K is chosen such that $e^{A_e\tau} - KC_{2e}$ is exponentially stable.

B. Proof:

See Appendix A.

III. Sample-Data Based Tracking Control for Magnetic Levitation System

Fig. 1 shows a schematic diagram of a magnetic levitation system where a ball of magnetic material is suspended by means of an electromagnet whose current is controlled by feedback from the optically measured ball position [14]. This system has the basic ingredients of systems constructed to levitate mass, used in gyroscopes, accelerometers, and fast trains. The equation for the motion of the ball is where $G = mg$ with m as the mass of the ball and $y \geq 0$ is the vertical position of the ball measured from a reference point. k is a viscous friction coefficient, g is the acceleration due to gravity, $F(y,i)$ given by (6) is the force generated by the electromagnet and I is it's electric current.

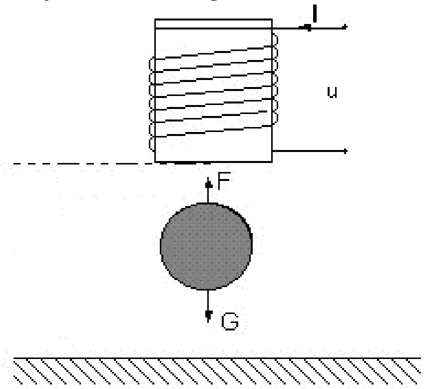

Fig. 1. Magnetic levitation system.

$$m\ddot{y} = -k\dot{y} + G + F(y, I) \quad (5)$$

$$F(y, I) = {-L_0 I^2}\Big/{2a(1 + y/a)^2} \quad (6)$$

where L_0 and a are positive constants. For the system (5) we consider control of vertical position of the ball y . We want it to track the reference signal given by (3). The system (5) is linearized and (3, 5-6) can be rewritten as:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_2u(t)\\ \dot{w}(t) &= 0_{1x1}, \qquad w(0) = y_d\\ z(t) &= C_1x(t) + D_{11}w(t)\\ y(t) &= C_2x(t),\end{aligned} \quad (7)$$

where $A = \begin{bmatrix}0 & 1\\ 2g/(a + y_{ss}) & -k/m\end{bmatrix}$, $B_2 = \begin{bmatrix}0\\ M\end{bmatrix}$, $C_1 = \begin{bmatrix}1 & 0\end{bmatrix}$, $D_{11} = -1$ and $C_2 = \begin{bmatrix}1 & 0\end{bmatrix}$. y_{ss} is the steady state output set at 0.05m and $M = {-1}\big/{(a + y_{ss})}\sqrt{{2L_0ag}/{m}}$. For the system (7) we set the sampling time $\tau = 0.01$sec and assume the control input realized through a zero-order hold and the observation is taken at $k\tau$. The system (7) becomes

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_2\tilde{u}(t)\\ \dot{w}(t) &= 0_{1x1}, \qquad w(0) = y_d\\ z(t) &= C_1x(t) + D_{11}w(t).\end{aligned} \quad (8)$$

For this system we shall design an output feedback controller. To do so we first investigate if the assumptions A1-A3 are satisfied. It is obvious that A1 is satisfied. Since B_1 =0 and D_{12} =0 in the discrete regulator equations of A2, the solution of Π and Γ is:

$$\Pi = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \Gamma = \begin{bmatrix}44.3 & -1\end{bmatrix}$$

This satisfies A2. Assumption A3 is also satisfied if the detectability of pair ($C_{2e,}A_e$) is ensured which is achieved by choosing the sampling time to be nonpathological [9]. Thus, by applying Theorem 2.1 we can design an output feedback controller for the tracking control of (5). In this case all the eigenvalues of A and A_e are real and hence sampling time τ =0.01sec is nonpathological. To solve the tracking problem, it remains to design a feedback gain F and observer gain K. The feedback gain F is obtained by placing the eigen values at $\begin{bmatrix} e^{-0.8\tau} & e^{-0.8\tau} \end{bmatrix}$ resulting in

$$F = \begin{bmatrix} -44.44 & -0.53 \end{bmatrix}$$

This satisfies $\sigma(A_d + B_{2d}F) \subset D$. While observer gain K is obtained by placing the eigen values at $\begin{bmatrix} e^{-0.8\tau} & e^{-0.8\tau} & e^{-0.8\tau} \end{bmatrix}$ resulting in

$$K = \begin{bmatrix} 0.04 & 2.012 & 0.0 \end{bmatrix}^T$$

This satisfies $\sigma(e^{A_e\tau} - KC_{2e}) \subset D$. We have applied the proposed discrete-time controller (3) with the proposed controller and observer gains to the tracking control of vertical position y of a magnetic suspension system (5). Here, we want the output to track the reference signal $w(0) = 10$. For, comparison, we have also applied the continuous-time controller (9) on the linearized system (5).

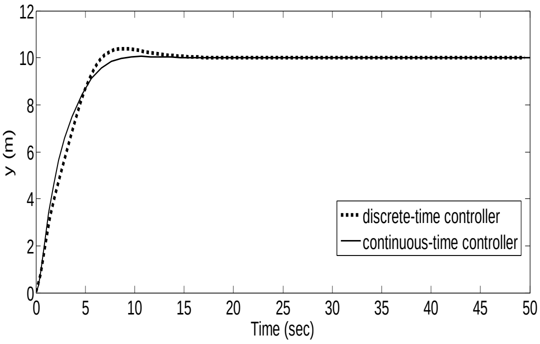

Fig. 2. The trajectories of y by using discrete and continuous controller

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = A_e \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}(t) + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} u(t) + K_1\{y(t) - C_{2e} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}(t)\}$$

$$u(t) \quad = \begin{bmatrix} F_1 & \Gamma - F_1\Pi \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}(t) \qquad (9)$$

The gain F_1 is obtained by placing eigenvalues at $\begin{bmatrix} -0.8 & -0.8 \end{bmatrix}$ and K_1 is designed for eigenvalues $\begin{bmatrix} -0.8 & -0.8 & -0.8 \end{bmatrix}$ Let the initial conditions of the system be:

$$x(0) = [0.05 \quad 0]^T, \qquad w(0) = [10]$$

and the initial conditions of the observed states be :

$$\hat{x}(0) = [0.5x(0)]^T, \qquad \hat{w}(0) = [0]$$

Simulations were done using Matlab and Simulink. The obtained results are shown in the figures below. Fig.2 shows the time response of the vertical position y with the controller (3) and (18). Error plots for both controllers are shown in Fig. 3 which shows that $\lim_{t\to\infty} z(t) = 0.$ It can be seen from the Fig. 2 and Fig. 3, the time response of the trajectories of the vertical position of the linearized magnetic suspension system is almost same for both discrete and continuous-time controllers. In both the controllers the tracking is achieved by the designed controllers.

IV. Conclusion

In this paper the output regulation of LTI sampled-data system and a constant exosystem is considered. The proposed discrete-time controller is designed by assuming that the regulator equations in discrete-time has a solution which led to designing the overall system in discrete domain realized through a zero-order hold. As an application the controller is applied to the tracking of the vertical positioon of a linearized magnetic levitation system and the comparison of the proposed controller with the continuous-time controller is presented.

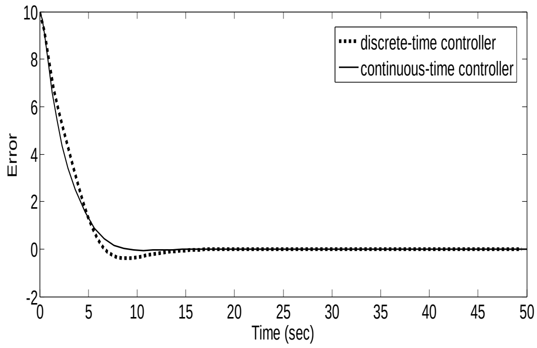

Fig. 3. Error plots for both discrete and continuous time controllers.

V. Appendix

A. Proof:

Since we have

$\tilde{u}(t) = u(t), k\tau \le t\left(k+1\right)\tau$ and $w(t) = w(k\tau) = w(0)$,

the sampled-data system (1) is rewritten as

$$\begin{aligned} x(k\tau + h) &= e^{Ah}x(k\tau) + \int_{k\tau}^{k\tau+h} e^{A(k\tau+h-r)}dr \\ &\quad \times\left[B_1w(0) + B_2u(k)\right] \\ &= e^{Ah}x(k\tau) + \Phi(h)\left[B_1w(0) + B_2u(k)\right] \\ z(k\tau + h) &= C_1x(k\tau + h) + D_{11}w(0) + D_{12}u(k), \\ y(k) &= C_2x(k\tau) + D_{21}w(0). \end{aligned}$$

for any $0 < h < \tau$ and $x(k\tau)$ is determined by the following discrete-time system

$$\tilde{x}(k+1) = A_d\tilde{x}(k) + B_{1d}\tilde{w}(k) + B_{2d}u(k)$$

where $\tilde{x}[k] = x(k\tau),\ \tilde{w}[k] = w(k\tau) = w(0)$ and

$$\Phi(h) = \int_0^h e^{A(h-s)}ds$$

Since

$$e^{A_e\tau} = \begin{bmatrix} A_d & B_{1d} \\ 0 & I \end{bmatrix}$$

The controller (3) can be re-written as

$$\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}[k+1] = \begin{bmatrix} A_d & B_{1d} \\ 0 & I \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}[k] + \begin{bmatrix} B_{2d} \\ 0 \end{bmatrix}u[k]$$

$$K\{y[k] - C_{2e}\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}[k]\}$$

$$u[k] \quad = \begin{bmatrix} F & \Gamma - F\Pi \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}$$

Let $e_1(k) = \tilde{x}(k) - \hat{x}(k),\ e_2(k) = \tilde{w}(k) - \hat{w}(k)$ and $e(k) = [e_1^T(k) \quad e_2^T(k)]^T$. Then we have

$$e(k+1) = \left\{\begin{bmatrix} A_d & B_{1d} \\ 0 & I \end{bmatrix} - K\{y(k) - C_{2e}\right\}e(k)$$

And by assumption we have $e(k) \to 0$ as $k \to \infty$. So,

$$\begin{aligned} x(k\tau+h) &= e^{Ah}x(k\tau) + \Phi(h)\{B_2Fx(k\tau) \\ &\quad + \left[B_1 + B_2\left(\Gamma - F\Pi\right)w(0)\right]\} \\ &\quad - \Phi(h)B_2\left[F \quad \Gamma - F\Pi\right]e(k) \end{aligned} \tag{10}$$

$$\begin{aligned} z(k\tau+h) &= C_1x(k\tau+h) + D_{12}Fx(k\tau) + \\ &\quad \left[D_{11} + D_{12}\left(\Gamma - F\Pi\right)\right]w(0) \\ &\quad - D_{12}\left[F \quad \Gamma - F\Pi\right]e(k) \end{aligned} \tag{11}$$

And

$$\begin{aligned} \tilde{x}(k+1) &= (A_d + B_{2d}F)\tilde{x}(k) + \\ &\quad [B_{1d} + B_{2d}\left(\Gamma - F\Pi\right)]\tilde{w}(k) \\ &\quad - B_{2d}\left[F \quad \Gamma - F\Pi\right]e(k) \end{aligned} \tag{12}$$

Since $A_d + B_{2d}F = f_\tau(A)(A + B_2F) + I$ and $B_{1d} + B_{2d}(\Gamma - F\Pi) = f_\tau(A)[B_1 + B_2(\Gamma - F\Pi)],$ (12) becomes

$$\begin{aligned} \tilde{x}(k+1) &= [f_\tau(A)(A + B_2F) + I]\tilde{x}(k) + \\ &\quad f_\tau(A)[B_1 + B_2(\Gamma - F\Pi)]\tilde{w}(k) \\ &\quad - B_{2d}\left[F \quad \Gamma - F\Pi\right]e(k) \end{aligned}$$

We know that $\tilde{w}[k] = w(0)$ for any k, $\sigma(A_d, B_{2d}) \subset D$ and $x_\infty = \lim_{k\to\infty}\tilde{x}(k)$, we have

$$\begin{aligned} x_\infty &= [f_\tau(A)(A + B_2F) + I]x_\infty + \\ &\quad f_\tau(A)[B_1 + B_2(\Gamma - F\Pi)]w(0) \\ 0 &= [f_\tau(A)(A + B_2F) + I]x_\infty + \\ &\quad f_\tau(A)[B_1 + B_2(\Gamma - F\Pi)]w(0) \end{aligned}$$

By the stabilizability of (A_d, B_{2d}), $f_\tau(A)$ is invertible so,

$$(A + B_2F)x_\infty + [B_1 + B_2(\Gamma - F\Pi)]w(0) = 0$$

Using A2**,** we have

$$\Pi S_d = A_d\Pi + B_1d + B_{2d}\Gamma$$

$$\Pi I \ = (f_\tau(A)A + I)\Pi + f_\tau(A)B_2\Gamma + f_\tau(A)B_1$$

$$0 \quad = A + B_2\Gamma + B_1$$

where $B_1 + B_2(\Gamma - F\Pi) = -(A + B_2F)\Pi$ hence $(A + B_2F)[x_\infty - \Pi w(0)] = 0$ .Since $\sigma(A + B_2F) \subset C^-$ which implies that $A + B_2F$ is invertible and we have

$$x_\infty = \Pi w(0) \tag{13}$$

From (10) we get

$$\begin{aligned} x(k\tau+h) &= \Phi(h)(A + B_2F)[x(k\tau) - \Pi w(0)] + x(k\tau) \\ &\quad - \Phi(h)B_2[F \ \ \Gamma - F\Pi]e(k) \end{aligned}$$

Using (13) we get

$$\begin{aligned} \lim_{k\to\infty} x(k\tau+h) &= \Phi(h)(A + B_2F)\lim_{k\to\infty}[x(k\tau) - \Pi w(0)] \\ &\quad + \lim_{k\to\infty} x(k\tau) \\ &= x_\infty \text{ for any } 0 \le h < \tau \end{aligned} \tag{14}$$

By, (13), (14) and $\lim_{t\to\infty} e(t) = 0$ we can ensure that the closed-loop system (1) and (3) is asymptotically stable since $\lim_{t\to\infty} x(t) = 0$ and hence $\lim_{t\to\infty}\hat{x}(t) = 0$ if $w(t) = w(0)$. Using A2 we have

$$0 \quad = C_{1d}\Pi + D_{11d} + D_{12d}\Gamma \tag{15}$$

where step invariant transformation maps $\left(A, B, C, D\right) \leftrightarrow \left(A_d, B_d, C, D\right)$, thus (15) becomes,

$$0 \quad = C_1\Pi + D_{11} + D_{12}\Gamma \tag{16}$$

Now, using (11), (14) and (16) we have

$$\lim_{k\to\infty} z(k\tau+h) = \lim_{k\to\infty}\{C_1x(k\tau+h)+D_{12}Fx(k\tau)+ \left[D_{11}+D_{12}\left(\Gamma-F\Pi\right)\right]w(0)\}$$
$$=(C_1\Pi+D_{11}+D_{12}\Gamma)w(0)$$
$$=0 \quad \text{for any } 0\le h<\tau$$

This implies $\lim_{t\to\infty} z(t)=0$ and hence we have the assertion.

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Muwahida Liaquat received the BE degree in computer system engineering in 2004 and MS degree in electrical engineering in 2006 and is currently pursuing the PhD degree from College of Electrical and Mechanical Engineering, National University of Sciences and Technology, Rawalpindi, Pakistan. Her research interests are in image processing and regulation control of sampled data impulsive systems.

Mohammad Bilal Malik received the B.S. degree in Electrical Engineering from College of Electrical and Mechanical Engineering (EandME), Rawalpindi, Pakistan, in 1991. He received his M.S. degree in Electrical Engineering from Michigan State University (MSU), Michigan, USA in 2001. In 2004, he received his PhD degree in Electrical Engineering from National University of Sciences and Technology (NUST), Rawalpindi, Pakistan. He has been teaching at College of EandME, National University of Sciences and Technology (NUST) since 1991. His research focuses on signal processing and adaptive filtering for communications and control systems.