Exact Computation of the Triangular-Lattice Ising Model with Eighteen Spins on a Side

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***Abstract*—The Ising model, consisting of magnetic spins, is the most important system in understanding phase transitions and critical phenomena. For the first time, the exact integer values for the density of states of the triangular-lattice Ising model with eighteen spins on a side and free boundary conditions are evaluated. Also, the exact specific heats are obtained for the triangular-lattice Ising ferromagnet and antiferromagnet at the same time.**

***Index Terms*—Exact computation, triangular-lattice Ising model, density of states.**

I. Introduction

Phase transitions and critical phenomena are the most universal phenomena in nature. The Ising model, consisting of magnetic spins, is the simplest system showing phase transitions and critical phenomena at finite temperatures. The Ising model has played a central role in our understanding of phase transitions and critical phenomena [1]. Also, the Ising model explains the gas-liquid phase transitions accurately. Based on the Ising model, various theoretical methods such as mean-field theory, power-series expansion and analysis, renormalization group, and canonical transfer matrix have been developed to understand phase transitions and critical phenomena.

In particular, computer simulations have been the most popular method in studying phase transitions and critical phenomena due to the recent fast growth of computer hardware and software technologies. To investigate the phase transition and critical behavior of a given system as a continuous function of temperature, to obtain the partition function zeros showing most effectively phase transitions and critical phenomena, and to perform microcanonical analysis for phase transitions and critical phenomena, we need to calculate the density of states as a function of energy. The most computational methods to calculate the density of states yield the approximate density of state [2]-[13].

On the other hand, the microcanonical transfer matrix [14]-[39] is an exact computation method to calculate the exact integer values for the density of states. Until now, the exact integer values for the density of states of the Ising model on equilateral triangular lattice with free boundary conditions have been obtained up to fifteen spins on a side (corresponding to $2^{120} \approx 1.3\times10^{36}$ states) [15]. In this work, for the first time, we evaluate the exact integer values for the density of states of the triangular-lattice Ising model with eighteen spins on a side (corresponding to $2^{171} \approx 3.0\times10^{51}$ states). It is very difficult task to classify all 2^{171} spin configurations according to their energy values.

Using the exact density of states of the triangular-lattice Ising model with eighteen spins on a side, we obtain much more accurately the specific heats of two different systems (the triangular-lattice Ising ferromagnet and antiferromagnet) at the same time. Based on the specific heats of the triangular-lattice Ising ferromagnet and antiferromagnet, we discuss the phase transitions and critical phenomena of these systems.

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II. Ising Model

The Ising model on the equilateral triangular lattice [15] with eighteen spins on a side and free boundary conditions is defined by the Hamiltonian

$$H = J\sum_{<i,j>}(1-\sigma_i\sigma_j),$$

where J is the coupling constant, $<i, j>$ indicates a sum over all bonds between any nearest-neighbor spin pairs σ_i and σ_j, and $\sigma_i=\pm1$ (1 for upward magnetic spin and –1 for downward magnetic spin). On triangular lattice, each spin has the six nearest neighbor spins except for the spins on the boundary edges. The triangular-lattice Ising model with L spins on a side has $N=L(L+1)/2$ spins and $B=3(N–L)$ bonds. Therefore, there are N=171 spins and B=459 bonds for the equilateral triangular lattice with eighteen spins on a side (L=18) and free boundary conditions.

Next, we define the density of states, $\Omega(E)$, with a given energy

$$E = \sum_{<i,j>}(1-\sigma_i\sigma_j),$$

where E is integers between 0 and $4B/3$=612 for L=18. Then, the partition function of the triangular-lattice Ising model (a sum over all possible spin configurations)

$$Z = \sum_{\{\sigma\}}\exp(-\beta H),$$

where $\beta=1/kT$ (k is the Boltzmann constant and T is temperature), can be written as

$$Z(T) = \sum_{E=0}^{612}\Omega(E)\exp(-\beta JE).$$

The partition function is the most important function in thermodynamics, statistical mechanics, and physical chemistry.

III. Density of States

The microcanonical transfer matrix [14]-[39], an exact computation method, is applied to calculate the exact integer values for the density of states of the Ising model on the equilateral triangular lattice with eighteen spins on a side (L=18, N=171, B=459). First, an array $\omega^{(1)}$, which is indexed by energy E and the eighteen spin variables $\sigma_i^{(1)} (1 \le i \le 18)$ for the first row, is initialized with the seventeen horizontal bonds as

$$\omega^{(1)}(E;\sigma^{(1)}) = \delta(E + \sum_{n=1}^{17} \sigma_n^{(1)} \sigma_{n+1}^{(1)} - 17),$$

where δ is the Kronecker delta. Second, by introducing the seventeen spin variables $\sigma_i^{(2)} (1 \le i \le 17)$ for the second row and considering the thirty-four vertical bonds between the first and second rows, the array $\omega^{(1)}$ is modified into

$$\varpi^{(2)}(E;\sigma^{(2)}) = \omega^{(1)}(E + \sum_{n=1}^{17} \sigma_n^{(2)} (\sigma_n^{(1)} + \sigma_{n+1}^{(1)}) - 34; \sigma^{(1)}).$$

After taking the sixteen horizontal bonds of the second-row spins, we obtain

$$\omega^{(2)}(E;\sigma^{(2)}) = \varpi^{(2)}(E + \sum_{n=1}^{16} \sigma_n^{(2)} \sigma_{n+1}^{(2)} - 16; \sigma^{(2)}).$$

Next, for the third row, if we introduce the sixteen spin variables $\sigma_i^{(3)} (1 \le i \le 16)$ and consider the thirty-two vertical bonds, we have

$$\varpi^{(3)}(E;\sigma^{(3)}) = \omega^{(2)}(E + \sum_{n=1}^{16} \sigma_n^{(3)} (\sigma_n^{(2)} + \sigma_{n+1}^{(2)}) - 32; \sigma^{(2)}).$$

Now, the fifteen horizontal bonds connecting the spins in the third row are taken into account by shifting the energy:

$$\omega^{(3)}(E;\sigma^{(3)}) = \varpi^{(3)}(E + \sum_{n=1}^{15} \sigma_n^{(3)} \sigma_{n+1}^{(3)} - 15; \sigma^{(3)}).$$

After repeating these steps, the final spin $\sigma_1^{(18)}$ in the eighteenth row is introduced with the two vertical bonds such as

$$\varpi^{(18)}(E;\sigma^{(18)}) = \omega^{(17)}(E + \sigma_1^{(18)} (\sigma_1^{(17)} + \sigma_2^{(17)}) - 2; \sigma^{(17)}).$$

Finally, the exact integer values for the density of states of the triangular-lattice Ising model with eighteen spins on a side is given by

$$\Omega(E) = \sum_{\sigma} \varpi^{(18)}(E;\sigma_1^{(18)}),$$

as shown in Table I, II, and III. The sum over all densities of states is exactly equal to the number of all possible spin configurations:

$$\sum_E \Omega(E) = 2^{171} \approx 3.0 \times 10^{51}.$$

TABLE I: Exact Integer Values for the Density of States $\Omega(E)$ of the Ising Model on the Equilateral Triangular Lattice with Eighteen Spins on a Side and Free Boundary Conditions, as a Function of Energy E (=0~248)

E	$\Omega(E)$
0	2
4	6
8	120
12	698
16	4926
20	32898
24	183598
28	1077360
32	5732928
36	29742908
40	149995056
44	725682150
48	3441025398
52	15850948398
56	71378421714
60	314842377450
64	1360270736442
68	5773198112370
72	24079429379534
76	98835422837142
80	399631763275344
84	1593020943584184
88	6266017974032538
92	24336615611053698
96	93394373030137502
100	354349069960204254
104	1329908486113610586
108	4939747629298345624
112	18166324908534247938
116	66172517125822650006
120	238828273199602595280
124	854324137189574254854
128	3029724735335329863798
132	10654321072330846416222
136	37159871767171454952942
140	128563609273897927138074
144	441279588731867009146194
148	1502816275339314440210718
152	5078404212126656898914814
156	17029451373598977602838314
160	56668176518722621470880116
164	187131754821406286835976524
168	613227992362498144545682982
172	1994123411279023994947380294
176	6434584028140782226639532112
180	20601752121939915250005359616
184	65444705313464469115525800978
188	206251950660318896085506890758
192	644817488924589508544194155446
196	1999625296622038907636275808550
200	6150176507800464864457145926374
204	18758764707707840199874164120552
208	56734322502598178279959938358008
212	170120524893135900490975462542828
216	505682708225017649369928170719504
220	1489870977644177452447531651457436
224	4350151776353064838774025656713900
228	12585743860761393660220709604901732
232	36074742254773441282921967499712296
236	102425068949841268490650936689064146
240	288014606367086068382310243413145754
244	801959230924593393823151339228959818
248	2210760545258121573655221760353521826

The ferromagnetic ($J > 0$) ground states correspond to

$$\Omega(E=0)=2,$$

and the antiferromagnetic (J < 0) ground states are quite degenerate such as

$$\Omega(E = 612) = 23665003296449525435806996826,$$

corresponding approximately to 2.4×10^{28}.

The largest density of states is

$$\Omega(E = 460) = 2233447426155287986252910619652994403063548408934 12,$$

corresponding approximately to 2.2×10^{50}. This kind of a large integer number is stored in a computer by using a positional numeral system with a radix (or base) of 2^{31} such as

$$\Omega(E=460)=\sum_j P_j(2^{31})^{j-1}.$$

Here, we need the six P_j's as follows:

$$P_1 = 143620068,$$

$$P_2 = 1321816796,$$

$$P_3 = 1227133195,$$

$$P_4 = 1023651300,$$

$$P_5 = 424290496,$$

and

$$P_6 = 4890.$$

It should be noted that the exact integer values for the density of states of the Ising model on the equilateral triangular lattice with eighteen spins on a side and free boundary conditions are obtained for the first time. Even the approximate values for the density of states of the triangular-lattice Ising model with eighteen spins on a side and free boundary conditions have never been calculated by using other non-exact methods.

IV. Exact Specific Heats

Given the exact integer values for the density of states $\Omega(E)$, the free energy F is exactly given by

$$F=-kT\ln Z(T).$$

From the exact free energy, the exact thermodynamic functions can be obtained. For example, the exact specific heat can be expressed as [32], [37]

$$C(T)=(NkT^2)^{-1}\frac{\partial^2}{\partial\beta^2}\ln Z(T).$$

It should be noted that the exact specific heats of two different systems (the ferromagnet and the antiferromagnet) can be obtained at the same time in this work.

Fig. 1 shows the exact specific heat of the Ising ferromagnet (J>0) on the equilateral triangular lattice with eighteen spins on a side and free boundary conditions. The specific heat shows the sharp peak at T=3.232J/k, signaling the phase transition between the low-temperature ferromagnetic phase and the high-temperature paramagnetic phase.

TABLE II: Exact Integer Values for the Density of States $\Omega(E)$ of the Ising Model on the Equilateral Triangular Lattice with Eighteen Spins on a Side and Free Boundary Conditions, as a Function of Energy E (=252~500)

E	$\Omega(E)$
252	603257429670220756799215961763105438 8
256	1629119553186725440968175301195957328 6
260	4353194741428652871674183176408074541 8
264	11507512657356658135840890715748895881 2
268	30087367865325231520996224850619721732 0
272	77790254711891750888635788559702637136 6
276	198843343978148396631213189360310154212 6
280	502395867336865560104898158574732292220 4
284	1254388524486744602565251913003250228513 2
288	3094343528116924837228742165227494916744 8
292	7539669798982548539261216487926224152534 0
296	18141691551349653780295896069199420782098 4
300	43095954996124543805756239461721001126480 8
304	101045591017755137799034081450618406995314 0
308	233779451073471774624432220443281622473445 6
312	533563857348253757349047796758933354639730 0
316	1200984698788262733399932376543193814515575 6
320	2665239390268442925069680354859561364732673 4
324	5829831073611025512330527435497823730306777 0
328	12565117026443633185692123735821345389606966 8
332	26676822074853844747658975424701125511446770 6
336	55772612036812062039572234479358088246598353 6
340	114785114770475688449386955888884004153357404 8
344	232478092992574532523529638049595552075068653 0
348	463191057846823028477076411961488177222210604 6
352	907538736279688333033622435470527601354971934 8
356	1747983892332715495051093438049264307536490088 8
360	3308356046405871641919968037581799893757068196 2
364	6150629952641750486495533562045918748295803154 8
368	11227520728497889011580057359186081888102377679 0
372	20115160493368722488781979221249202630049607748 2
376	35354991480581655355011067295220175148567334584 4
380	60935618479988761336270919993053523355409866660 6
384	102939883000162555975376200800372922264066652220 2
388	170364634224704245783424130654770665957756622821 8
392	276084578727240655557399611417120218529726023181 4
396	437871982758714309346021943364764184602710511849 8
400	679299242716310760733059138859858340931160403880 6
404	1030247282311950657255111479244252383159014901497 8
408	1526637019498035862570519088881579026643864242630 2
412	2208926261214300270424805181943968648635627646606 6
416	3118916963276662998579547469655408983160336269456 4
420	4294551287025124134967714976196119628573816437634 0
424	5762683635612467586313796117924521703010549412071 0
428	7530311302093999631654514318751313069589981481495 8
432	9575380634904084515444389376898704401705208916819 2
436	11838932941604347301141993046223330076409475777893 6
440	14220814117190504674091672351181358553299182733907 4
444	16581205836700676538405700764619307113674975098757 4
448	18749641882509601372651693052860948074858198363575 4
452	20541881702164023488273617872874380773000775217845 0
456	21783175570626180088069973415384365240070143046107 8
460	22334474261552879862529106196529944030635484089341 2
464	22116601540419879982187232033155857608483198165962 4
468	21126930915249281211948837485452570146276798529612 4
472	19444086482280708986831923536566664941287141118094 0
476	17218592990590990244243634816481646981859999494740 8
480	14650691284169554218762215550352996294379776903617 2
484	11959770900931035568484847685573333205165383528033 2
488	9352015995864723798499536950962638355995159832837 2
492	6993176681416250842155697772747065638014433283180 2
496	4991739300786926217455719139130521020831187074068 6
500	3394739138266760477440075408737348164774735428690 2

TABLE III: EXACT INTEGER VALUES FOR THE DENSITY OF STATES $\Omega(E)$ OF THE ISING MODEL ON THE EQUILATERAL TRIANGULAR LATTICE WITH EIGHTEEN SPINS ON A SIDE AND FREE BOUNDARY CONDITIONS, AS A FUNCTION OF ENERGY *E* (=504~612)

E	$\Omega(E)$
504	21950781726445057354293572646890974133910427062586
508	13465804322590765693109900843417171490205709454840
512	7818704538801627208348282883093631189281721542472
516	4286118920863464319012351239079820313385928372184
520	2212305652992360430441136002143791362484968188504
524	1072041653307650893550563631998246073083876491714
528	486181006450383850841607109660327484773133664052
532	205649417094675483217121122612034006852750305776
536	80835122735378429057361360565289896153212053658
540	29408820840369626431217743962720078903176877928
544	9859751964338024711704821054735749328800880912
548	3031755345218311633082572397197569224058917256
552	850526355685300635904139272904828377861799116
556	216439491775713945247246780347735395063388510
560	49642320030960240892240905384932124041386758
564	10188582054939574080136950772359492660736290
568	1856076608608365806023004991979821849783174
572	297351124582870094358418260738393802564494
576	41444399841439401286605731668575495441304
580	4962271488410140540704733439860177529052
584	502673946829119573918563783000154607602
588	42275793284432184726316549690456143940
592	2881565478570894451055004821430792480
596	154143255292560847456301000415328128
600	6182853767838614190348673645213992
604	173315951796170858726377113576180
608	2994826377002713783475651283510
612	23665003296449525435806996826

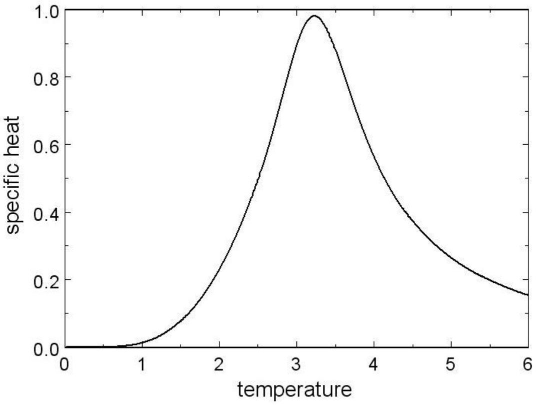

Fig. 1. Exact specific heat (in units of the Boltzmann constant *k*) per volume as a function of temperature (in units of *J*/*k*) for the triangular-lattice Ising ferromagnet (*J* > 0) with eighteen spins on a side.

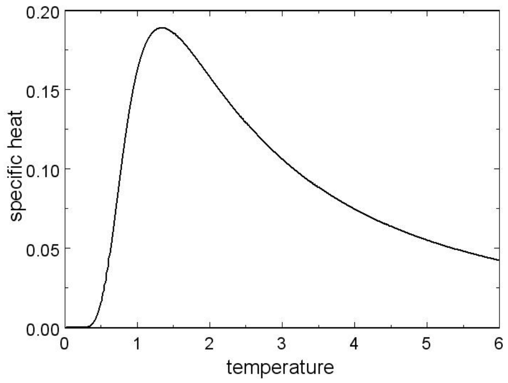

Fig. 2. Exact specific heat (in units of *k*) per volume as a function of temperature (in units of |*J*|/*k*) for the triangular-lattice Ising antiferromagnet (*J* < 0) with eighteen spins on a side.

Fig. 2 shows the exact specific heat of the Ising antiferromagnet (J<0) on the equilateral triangular lattice with eighteen spins on a side and free boundary conditions. The specific heat shows a peak at T=1.340|J|/k. But the peak of the triangular-lattice Ising antiferromagnet is not sharp, compared to the peak of the triangular-lattice Ising ferromagnet. Rather, the peak for the specific heat of the triangular-lattice Ising antiferromagnet resembles the Schottky-anomaly peak for the specific heat of the one-dimensional Ising model [40].

V. CONCLUSION

For the first time, we have evaluated the exact integer values for the density of states of the Ising model on the equilateral triangular lattice with eighteen spins on a side and free boundary conditions, by classifying all $2^{171} \approx 3.0\times10^{51}$ spin configurations according to their energy values. Using the exact density of states of the triangular-lattice Ising model with eighteen spins on a side, we have obtained much more accurately the specific heats of two different systems (the triangular-lattice Ising ferromagnet and antiferromagnet) at the same time. Based on the specific heats of the triangular-lattice Ising ferromagnet and antiferromagnet, we have investigated the phase transitions and critical phenomena of these systems. The specific heat of the triangular-lattice Ising ferromagnet has shown a sharp peak, signaling the phase transition between the low-temperature ferromagnetic phase and the high-temperature paramagnetic phase. On the other hand, the specific heat of the triangular-lattice Ising antiferromagnet has shown the Schottky-anomaly peak.

REFERENCES

[1] C. Domb, *The Critical Point*, London, U.K.: Taylor and Francis, 1996.

[2] G. Bhanot, R. Salvador, S. Black, P. Carter, and R. Toral, "Accurate estimate of ν for the three-dimensional Ising model from a numerical measurement of its partition function," *Physical Review Letters*, vol. 59, pp. 803-806, 1987.

[3] B. A. Berg and T. Neuhaus, "Multicanonical ensemble: A new approach to simulate first-order phase transitions," *Physical Review Letters*, vol. 68, pp. 9-12, 1992.

[4] J. Lee, "New Monte Carlo algorithm: Entropic sampling," *Physical Review Letters*, vol. 71, pp. 211-214, 1993.

[5] F. Wang and D. P. Landau, "Multiple-range random walk algorithm to calculate the density of states," *Physical Review Letters*, vol. 86, pp. 2050-2053, 2001.

[6] C.-O. Hwang, S.-Y. Kim, D. Kang, and J. M. Kim, "Ising antiferromagnets in a nonzero uniform magnetic field," *Journal of Statistical Mechanics*, vol. 7, pp. 1-8, 2007.

[7] S.-Y. Kim, C.-O. Hwang, and J. M. Kim, "Partition function zeros of the antiferromagnetic Ising model on triangular lattice in the complex temperature plane for nonzero magnetic field," *Nuclear Physics B*, vol. 805, pp. 441-450, 2008.

[8] C.-O. Hwang and S.-Y. Kim, "Yang-Lee zeros of triangular Ising antiferromagnets," *Physica A*, vol. 389, pp. 5650-5654, 2010.

[9] J. H. Lee, H. S. Song, J. M. Kim, and S.-Y. Kim, "Study of a square-lattice Ising superantiferromagnet using the Wang-Landau algorithm and partition function zeros," *Journal of Statistical Mechanics*, vol. 10, pp. 1-9, 2010.

[10] S.-Y. Kim, "Evaluation of Wang-Landau Monte Carlo simulations," *International Proceedings of Computer Science and Information Technology*, vol. 22, pp. 125-129, 2012.

[11] S.-Y. Kim, "Evaluation of an efficient Monte Carlo algorithm to calculate the density of states," *International Journal of Machine Learning and Computing*, vol. 2, pp. 144-149, 2012.

[12] S.-Y. Kim and W. Kwak, "Study of the antiferromagnetic Blume-Capel model by using the partition function zeros in the complex temperature plane," *Journal of the Korean Physical Society*, vol. 65, pp. 436-440, 2014.

[13] C.-O. Hwang and S.-Y. Kim, "Field-induced Kosterlitz-Thouless transition in critical triangular-lattice antiferromagnets," *Monte Carlo Methods and Applications*, vol. 20, pp. 217-221, 2014.

[14] G. Bhanot, "A numerical method to compute exactly the partition function with applications to $Z(n)$ theories in two dimensions," *Journal of Statistical Physics*, vol. 60, pp. 55-75, 1990.

[15] B. Stosic, S. Milosevic, and M. E. Stanley, "Exact results for the two-dimensional Ising model in a magnetic field: Tests of finite-size scaling theory," *Physical Review B*, vol. 41, pp. 11466-11478, 1990.

[16] R. J. Creswick, "Transfer matrix for the restricted canonical and microcanonical ensembles," *Physical Review E*, vol. 52, pp. R5735-R5738, 1995.

[17] R. J. Creswick and S.-Y. Kim, "Finite-size scaling of the density of zeros of the partition function in first- and second-order phase transitions," *Physical Review E*, vol. 56, pp. 2418-2422, 1997.

[18] S.-Y. Kim and R. J. Creswick, "Yang-Lee zeros of the *Q*-state Potts model in the complex magnetic field plane," *Physical Review Letters*, vol. 81, pp. 2000-2003, 1998.

[19] S.-Y. Kim and R. J. Creswick, "Fisher zeros of the *Q*-state Potts model in the complex temperature plane for nonzero external magnetic field," *Physical Review E*, vol. 58, pp. 7006-7012, 1998.

[20] R. J. Creswick and S.-Y. Kim, "Microcanonical transfer matrix study of the *Q*-state Potts model," *Computer Physics Communications*, vol. 121, pp. 26-29, 1999.

[21] S.-Y. Kim and R. J. Creswick, "Exact results for the zeros of the partition function of the Potts model on finite lattices," *Physica A*, vol. 281, pp. 252-261, 2000.

[22] S.-Y. Kim and R. J. Creswick, "Density of states, Potts zeros, and Fisher zeros of the *Q*-state Potts model for continuous *Q*," *Physical Review E*, vol. 63, article: 066107, pp. 1-12, 2001.

[23] S.-Y. Kim, "Partition function zeros of the Q-state Potts model on the simple-cubic lattice," *Nuclear Physics B*, vol. 637, pp. 409-426, 2002.

[24] S.-Y. Kim, "Density of the Fisher zeros for the three-state and four-state Potts models," *Physical Review E*, vol. 70, pp. 1-5, 2004.

[25] S.-Y. Kim, "Yang-Lee zeros of the antiferromagnetic Ising model," *Physical Review Letters*, vol. 93, pp. 1-4, 2004.

[26] S.-Y. Kim, "Fisher zeros of the Ising antiferromagnet in an arbitrary nonzero magnetic field plane," *Physical Review E*, vol. 71, pp. 1-4, 2005.

[27] S.-Y. Kim, "Density of Yang-Lee zeros and Yang-Lee edge singularity for the antiferromagnetic Ising model," *Nuclear Physics B*, vol. 705, pp. 504-520, 2005.

[28] S.-Y. Kim, "Honeycomb-lattice antiferromagnetic Ising model in a magnetic field," *Physics Letters A*, vol. 358, pp. 245-250, 2006.

[29] S.-Y. Kim, "Density of Yang-Lee zeros for the Ising ferromagnet," *Physical Review E*, vol. 74, pp. 1-7, 2006.

[30] M. E. Monroe and S.-Y. Kim, "Phase diagram and critical exponent ν for nearest-neighbor and next-nearest-neighbor interaction Ising model," *Physical Review E*, vol. 76, article: 021123, pp. 1-5, 2007.

[31] S.-Y. Kim, "Ground-state entropy of the square-lattice *Q*-state Potts antiferromagnet," *Journal of the Korean Physical Society*, vol. 52, pp. 551-556, 2008.

[32] S.-Y. Kim, "Specific heat of the square-lattice Ising antiferromagnet in a magnetic field," *Journal of Physical Studies*, vol. 13, pp. 1-3, 2009.

[33] S.-Y. Kim, "Partition function zeros of the square-lattice Ising model with nearest- and next-nearest-neighbor interactions," *Physical Review E*, vol. 81, pp. 1-7, 2010.

[34] S.-Y. Kim, "Partition function zeros of the honeycomb-lattice Ising antiferromagnet in the complex magnetic-field plane," *Physical Review E*, vol. 82, pp. 1-7, 2010.

[35] S.-Y. Kim, "Honeycomb-lattice Ising model in a nonzero magnetic field: Low-temperature series analysis and partition function zeros," *Journal of the Korean Physical Society*, vol. 56, pp. 1051-1054, 2010.

[36] S.-Y. Kim, "Yang-Lee edge singularity of the square-lattice Ising ferromagnet," *Journal of the Korean Physical Society*, vol. 59, pp. 2205-2208, 2011.

[37] S.-Y. Kim, "Specific heat and partition function zeros of the three-state Potts model," *Journal of the Korean Physical Society*, vol. 59, pp. 2980-2983, 2011.

[38] S.-Y. Kim, "Ising antiferromagnets on honeycomb and square lattices in the critical magnetic field," *Journal of the Korean Physical Society*, vol. 61, pp. 1950-1955, 2012.

[39] S.-Y. Kim, "Exact partition functions of the Ising model on $L \times L$ square lattices with free boundary conditions up to L=22," *Journal of the Korean Physical Society*, vol. 62, pp. 214-219, 2013.

[40] S.-Y. Kim, "Generalized Schottky anomaly," *Journal of the Korean Physical Society*, vol. 65, pp. 970-972, 2014.

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